Parker Zenith & Crown
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Frequency ladders and internal resonance

Design review to detect and avoid energy transfer between vibration modes, with special attention to the golden-ratio ladder

> **Parker Zenith & Crown** is an aerospace and deep-tech engineering consultancy based in Lafayette (Louisiana). We work entirely computationally: analysis, numerical simulation and design review based on the data provided by the client. This document presents a well-known problem of nonlinear dynamics from the literature, the preliminary results of our simulations and the review service we offer on that basis.

1 The problem: internal resonance between modes

In a linear vibrating system, each natural mode evolves independently. When there is nonlinearity —large displacements, joints, materials or geometry—, the modes become coupled. The classical theory of nonlinear oscillations establishes that this coupling becomes especially effective when the natural frequencies satisfy an *internal resonance* relation. For quadratic nonlinearities, the most relevant one is the triad condition:

$$\omega_k \approx \omega_i + \omega_j \quad \text{(and the particular case } \omega_k \approx 2\omega_i\text{)}$$

If the condition is met, the energy put into one mode can be transferred to others, with responses that a linear modal analysis does not predict. This is a known and documented rule (A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*, Wiley, 1979; A. H. Nayfeh, *Nonlinear Interactions*, Wiley, 2000). Our work is an **application** of that rule, not a discovery of our own.

Why the golden ladder is a textbook case

If a design distributes its frequencies along a geometric ladder $\omega_n = r^{n-1}\cdot\omega_1$ with golden ratio $r = \phi \approx 1.618$, the triad condition is met **exactly** at every step, because

$$\varphi^{n+1} = \varphi^n + \varphi^{n-1} \quad \Rightarrow \quad \omega_{n+2} = \omega_n + \omega_{n+1}$$

In the range of ratios from 1.50 to 1.75, ϕ is moreover the only geometric ratio with any exact triad among the first seven steps. It is therefore a ladder prone to energy transfer between modes, although —as will be shown— not the only one, nor necessarily the most exposed.

2 Simulation results PRELIMINARY

All the results in this section come from numerical simulation on idealised models. No experimental test has been carried out and no failure has been observed in a real structure.

2.1 Exploratory simulation

Seven oscillators with generic quadratic nonlinearity, exciting only the top mode, without damping and in a single configuration. With ratio ϕ, the maximum fraction of energy leaving the excited mode is about $\mathbf{1.1\cdot10^{-3}}$; with the other ratios between 1.50 and 1.75 (step 0.01) it is $\mathbf{\leq 10^{-5}}$, i.e. about 100 times smaller (figure 1).

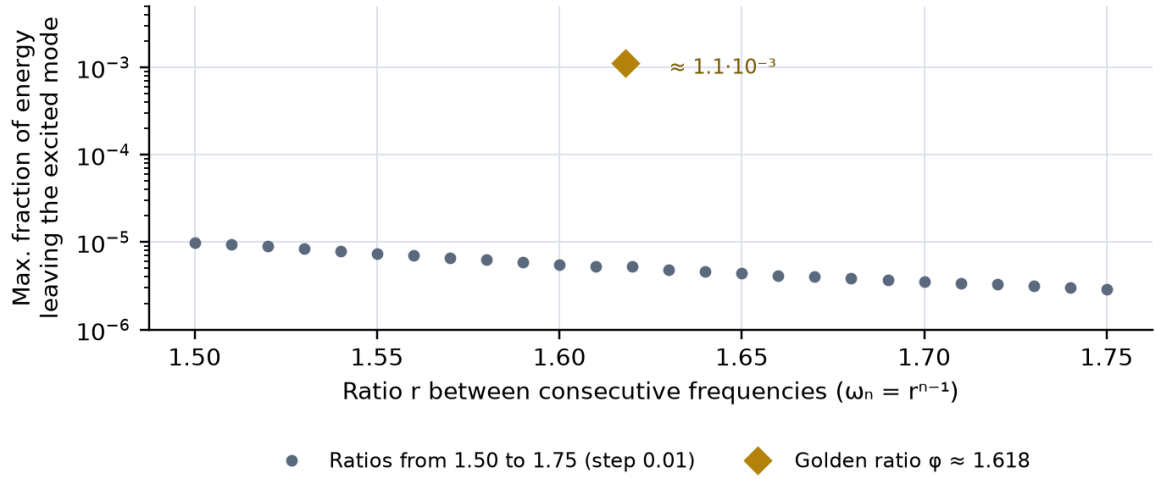


Figure 1. Exploratory simulation: 7 oscillators, quadratic nonlinearity, excitation of the top mode, no damping, a single configuration. Logarithmic vertical scale. Preliminary result.

2.2 Extended study with a pre-registered protocol

To check whether the effect holds under conditions closer to practice, the study was repeated with a protocol fixed in writing before simulating: 107 ratios between 1.50 and 1.75 (including ϕ and the nearby rationals 8/5, 13/8 and 5/3), several random coupling seeds, two excited modes, three weak excitation levels and modal damping from 0 (conservative reference) to 5 %, which covers the typical range cited by NASA standards and studies (e.g. NASA-STD-5002 takes 1 % per mode when there are no measurements). Summary:

Condition	Does the golden ladder stand out?
No damping (free vibration, 30 configurations)	Yes, in 21 of 30. The transfer with ϕ is, in median, about **63 times** that of the non-resonant ratios.
Against nearby rationals, no damping	Yes. Mean transfer percentile: ϕ 90; 13/8 68; 8/5 52; 5/3 46 (100 = the largest).
Forced response with damping from 0.3 % to 3 % (54 configurations)	Only in 2 of 54 (3.7 %), both with 0.3 % damping and strong triad coupling. At 1 % and 3 %, in none.
Control with cubic nonlinearity (no damping)	No (1 of 30).

Honest reading. In the undamped limit the golden ladder clearly transfers more energy, also compared with 8/5 and 13/8. With realistic damping and the weak excitation levels tested, the effect almost never activates: the hypothesis that the golden ladder is especially dangerous is **not confirmed** under those conditions. This agrees with classical theory: the transfer only appears when coupling and amplitude overcome damping and detuning. In addition, in some cases the transfer maximum appears shifted with respect to ϕ (because of the nonlinear frequency shift) or at other ratios with higher-order resonances (close to $\sqrt{3}$ and to $2^{2/3}$).

2.3 Real structures also have triads

We computed the analytical frequencies of reference structures (Euler–Bernoulli beams with different boundary conditions, simply supported rectangular plates, and strings or axial bars). None follows a geometric ladder, but many contain exact or nearly exact triads: the simply supported beam satisfies $\omega_5 = \omega_3 + \omega_4$ (since 9 + 16 = 25); the square plate has 5 exact triads among its first 7 distinct frequencies, and the string 12, against 5 for the 7-step golden ladder. The practical conclusion is that **the golden ladder is one prone configuration among others**, and that the risk must be assessed for each specific design, whatever its origin.

2.4 What is missing

- A suitable metric for damped free vibration (the one used saturates and does not discriminate), with a new pre-registered protocol.
- Higher excitation levels, closer to launch loads, more seeds and longer simulations.
- Nonlinear couplings estimated from finite-element models of real components (panels, booms).
- Experimental validation on a simple set-up tuned to a triad.

The five numerical control reruns with higher integration accuracy have been completed: all are within the energy drift limit and they do not change the results above.

3 The service: frequency-ladder review

We offer a design review service that **detects ladders and frequency combinations prone to internal resonance**, quadratic and cubic —including the golden ladder, but not only that one— and helps avoid them before they cause problems, at the stage when retuning a frequency is still cheap.

Input

- Natural or modal frequencies of the design (from a finite-element analysis, an analytical calculation or a modal test).
- Modal damping, if known.
- Optionally, the expected excitation range and the modes of interest.

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Output

- Frequency combinations that are candidates for internal resonance (golden ladders and quadratic and cubic combinations), with their detuning.
- Review priority for each candidate combination.
- Frequency retuning recommendations to move the design away from internal resonance conditions.

A frequency coincidence does not by itself imply risk: with realistic damping, many combinations never transfer energy. The priority indicates what to review first, not a measured risk. Preliminary results.

The assessment tool is proprietary and is provided as a service. The principle it relies on, internal resonance, is public knowledge from the theory of nonlinear oscillations; the internal details of the method are not published.

Application areas

- **Structural vibration** (main area): panels, deployable booms, instrument brackets and other light structures with lightly damped modes.
- **Areas to explore**, with no results yet: multitone test signals, telemetry, antennas, trajectories and aerospace metamaterials. All of them involve sets of frequencies in which relations of the type $\omega_k \approx \omega_i + \omega_j$ may be relevant; their practical interest is to be assessed case by case.
- **Out of scope:** flutter and other aeroelastic phenomena (coupling with the airflow).

4 Status and limitations

- The results come from **preliminary numerical simulation** on idealised models (oscillators in modal coordinates with generic couplings), not on specific structures.
- **Experimental validation is missing.** We do not claim to have observed any real failure caused by a frequency ladder.
- With realistic damping and weak excitation, the simulated effect is small or nil; its relevance depends on the amplitude, coupling and damping of each design.

- We check our work against the engineering literature on internal resonances and against published damping values. The service supports design review and does not replace analysis or qualification testing.

The company maintains a line of research in theoretical physics currently under verification; no results are published until it is validated.

References

1. A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*, Wiley, 1979.
2. A. H. Nayfeh, *Nonlinear Interactions: Analytical, Computational, and Experimental Methods*, Wiley, 2000.
3. NASA-STD-5002 Rev. B, *Structural Loads Analysis for Spacecraft and Payloads*.
4. NASA, technical reports NTRS 20180006315 and 20100038325 (damping values in space structures).

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